Lockout/tagout and optimal production control policies in failure-prone non-homogenous transfer lines with passive redundancy

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This paper considers a production problem for a transfer line subject to random failures and repairs, and differs from other studies on transfer lines. It considers a manufacturing system consisting of three machines (two machines with passive redundancy, and one in series with the previous ones) producing one part type. The control problem is subject to non-negative constraints on work-in-process (WIP). The decision variables are the production rates of two main machines and a standby machine, and influence the WIP levels, the inventory levels and the system's capacity, which is assumed to be described by a finite-state Markov chain. The objective of this paper is to minimise WIP and finished goods inventory costs; it also aims to respect the essential space–time during intervention on machine down, in order to minimise the possibility of the circumvention of protection devices or of the retraction of lockout/tagout procedures through a passive redundancy system. This paper therefore verifies the effect of passive redundancy on optimal stock levels. Given that an analytical or even a numerical solution of the problem is very difficult to find, and that we want to have a more realistic model for industries, we present a combined approach, which is presented based on a combination of analytical formalism, simulation modelling, design of experiments, and response surface methodology to optimise a transfer line with passive redundancy, producing one part type. The usefulness of the proposed approach is illustrated through a numerical example and a sensitivity analysis.

**Keywords:** production control; lockout/tagout; passive redundancy; simulation, experimental design; response surface methodology

### 1. Introduction

To meet increasing customer demands, and in the face of intense competition, companies must be able to produce good-quality products at lower cost. This requirement necessarily makes performance a major concern for all managers. To achieve this goal, we must ensure equipment availability, both functionally and in terms of optimal use. Production systems optimisation is a topic that is of interest to researchers as well as to industry. To increase the availability of machines at the functional level, several maintenance strategies have been developed (Kenne and Gharbi 1999, Kenne et al. 2003, Rezg et al. 2004). The problem of optimally controlling the production rates of a manufacturing system has been widely discussed in the scientific literature. For two decades, significant effort has been devoted to optimising production systems to meet their complexity, competition, and challenges of globalisation. However, despite the effectiveness of the techniques developed, other challenges, including control and risk management of adverse events during maintenance, still lie ahead.

Studies conducted in Quebec by the Institut de recherche Robert-Sauvé en santé et sécurité du travail (IRSST) (Chinniah and Champoux 2008) showed that in 2005, dangerous machinery led to the deaths of about 20 workers in Quebec and that there were 13,000 accidents linked to them. These accidents also caused $70 million in damages settled by the Commission de la santé et de la sécurité du travail (CSST). Another study on the safety of lockout/tagout of Sawmill equipment in Quebec conducted by the Institut de recherche Robert-Sauvé en santé et sécurité du travail (IRSST) (Giraud et al. 2008) led the occupational health and safety (OHS) community to recognise that nearly a quarter of all accidents occur during interventions by workers on machines that are down. Manufacturing systems operate in a stochastic environment because machines are subject to random breakdowns and repairs, and in addition, demand for finished goods may vary. It is possible to predict and control certain events while others occur randomly, and are beyond the control of manufacturing systems (Gershwin 2002). Production systems' dynamics degrade with the number of breakdowns and repairs, and with a rise in the number of breakdowns and...
repairs comes a reduction in the availability of machines, as well as an increase in the number of occupational hazards associated with maintenance activities; further, they can lead to serious machine-related accidents, which are costly in terms of human life, sick-leave days, and general financial costs. An important question arises: how can an optimal production policy be maintained in an uncertain environment, while increasing the safety of maintenance workers? The preferred method used to overcome these problems is the lockout/tagout. It involves locking a machine with a padlock in order to discharge all sources of residual energy (hydraulic, electrical, etc.) in order to avoid premature starting of equipment throughout a maintenance intervention.

Many managers and workers wrongly believe it takes too long to plan and carry out lockout/tagout, thinking the accompanying downtime reduces productivity or performance. Consequently, lockout/tagout is often deficient because idle production time is seen as a hurdle to planned production rates. In this view, Charlot et al. (2006) considered an analytical model combining lockout/tagout, production and corrective maintenance policies for a single machine producing one part type. This work showed that lockout/tagout time can be better controlled with the proper scheduling in production plan control. Another work integrating lockout/tagout into operational risk in production control is proposed by Emami-Mehrgani et al. (2011). They considered a manufacturing system with passive redundancy consisting of two non-identical machines. Their work demonstrates clearly that passive redundancy optimises production and maintenance costs while enhancing occupational safety. Even greater benefits occur if effective lockout/tagout maintenance planning occurs in concert with production control.

Many authors have contributed to solving manufacturing-systems production-planning problems. Based on Rishel’s (1975) formulation, Older and Suri (1980) devised manufacturing systems with unreliable machine control problems. In their model, breakdowns and repairs are described by a homogeneous Markov process. The main difficulty with this approach is its lack of efficient methods for solving the optimisation problem characterised by Hamilton–Jacobi–Bellman (HJB) equations. Akella and Kumar (1986) analytically solved a one-machine, one part-type problem. In this view, Lou and Zhang (1994) extended the problem in Akella and Kumar (1986) to a flow control problem for a tandem production system consisting of two unreliable machines and conducted a rigorous study of the dynamic properties of the system. Similarly, Presman et al. (1995) considered a production-planning problem in an N-machine flow-shop system subject to machine breakdowns and repairs and to non-negativity constraints on work-in-process. Based on the formulation presented in Presman et al. (1995), Hajji et al. (2009) studied production and changeover control production for a buffered flow-shop producing several types of parts. Hajji et al. (2009) have developed dynamic programming equations in terms of problem directional derivative (DPEDD) and adopted a numerical approach to solve them. The purpose of this paper is to control the production rate of the machines in a transfer line with passive redundancy, which is one of the most important structures in reliability engineering and has been widely used in manufacturing systems. The main contributions of this work are divided into two parts: reduce production costs and provide free space–time to minimise the possibility of circumvention of protection devices or retraction of lockout/tagout procedures for machines under repair. These goals are reached for a transfer line through a passive redundancy system.

In this paper, by making use of the fact that the value function is the unique solution for the associated HJB equations, in terms of directional derivatives (DD), the structure of the solution under appropriate conditions is obtained. Given that an analytical or even a numerical solution of the problem is very difficult to find and that we want to have a more realistic model for industries, we present in this paper an alternative procedure based on the combination of analytical control approach and the experimental design method based on simulation experiments to find an approximation of the optimal control policy. A simulation-based experimental design approach is combined with the control theory to develop a systematic control approach as in Gharbi and Kenneé (2000) in the case of a production line with passive redundancy. The proposed control approach involves estimating the relationship between the incurred cost and the parameters of the control policy considered in this paper as control factors. The hedging point policy, parameterised by these factors, is used to conduct simulation experiments. For each configuration of input factors values, the simulations model is used to determine the related output (incurred cost). An input–output data set is then generated by a simulation model. The significant effects of input factors are determined by experimental design, and a response surface methodology is used to obtain the relationship between the input and the output factors in order to estimate the cost function. The best values for control factors are determined through this relationship.

This article is organised as follows. Assumptions and notations are defined in the next section. In Section 3, we provide the problem statement. Section 4 provides a numerical example and a sensitivity analysis, and the related production policy is presented. In Section 5, we present the control approach, the experimental design, and the response surface methodology. Finally, we conclude the paper in Section 6.
2. Assumptions and notations
This paper incorporates the following assumptions and notations:

2.1 Assumptions
(1) Corrective maintenance is carried out with lockout/tagout.
(2) The main machine is more robust than the standby machine.
(3) The main machine and the standby machine produce the same type of parts for the work-in-process (WIP).
(4) The main machine returns to production immediately after each repair (corrective maintenance with lockout/tagout), and the standby machine stands idle.

Assumption 4 is a classical assumption with a passive redundancy system and is due to the nature of a passive redundancy system.

2.2 Notations
The following notations are used in the rest of this article:

- $x_{11}()$: inventory level of work-in-process
- $x_{21}()$: inventory/backlog level of finished product
- $c^+_1$: holding cost incurred on buffer
- $c^+_2$: holding cost incurred on finished product
- $c^-_2$: backlog cost incurred on finished product
- $c^a$: cost incurred for the operation of the machine under repair at mode $a$
- $c_{r_1}$: corrective maintenance cost of main machine $M_1$
- $c_{r_2}$: corrective maintenance cost of main machine $M_2$
- $c_{r_s}$: corrective maintenance cost of standby machine $M_s$
- $c_{tagout}$: lockout/tagout cost
- $u_{r_1}$: corrective maintenance rate with lockout/tagout of main machine $M_1$
- $u_{r_2}$: corrective maintenance rate with lockout/tagout of main machine $M_2$
- $u_{r_s}$: corrective maintenance rate with lockout/tagout of standby machine $M_s$
- $g()$: instantaneous cost
- $J()$: total cost
- $v()$: value function
- $\rho$: discount rate
- $d$: demand rate
- $u_i()$: production rate of machine $i$ ($i = 1, 2, s$)
- $u_i^{max}()$: machine's $i$ ($i = 1, 2, s$) maximal production rate
- $q_{12}$: main machines $M_1$ and $M_2$ failure rate
- $q_{1s}$: standby machine $M_s$ failure rate

3. Problem statement
In this paper, we consider a flow control problem for a tandem production system with passive redundancy consisting of three unreliable machines. Two machines are in series ($M_1$ and $M_2$), and one machine is in standby with the main machine $M_1$, namely $M_s$ (standby machine). Recall that in this paper, by using the fact that the value function is the unique solution for the associated HJB equations, in terms of DD, the structure of the solution under appropriate conditions is obtained. Since either an analytical, a numerical solution of this problem or even an explicit functional relationship between the independent variables of the model (stock level) and performance criteria (cost incurred) is not usually available, an alternative procedure based on the combination of analytical control approach and the experimental design method based on simulation experiments is presented in this paper. A parameterised near-optimal control policy is used in the proposed control approach as input for the simulation model. In order to propose an approach that could be easily applied to control manufacturing systems for a tandem
production system with passive redundancy. The manufacturing system consists of three unreliable machines (two machines with passive redundancy, and one in series with the previous ones) at the operational level, and the descriptive capacities of discrete event simulations models are combined with analytical models, experimental design, and response surface methodology. The system is shown in Figure 1. The main machines have two states: $\xi_{i,2}(t) = 1$ if the main machine $M_i$ ($i = 1, 2$) is operational and $\xi_{i,2}(t) = 2$ if the main machine $M_i$ ($i = 1, 2$) is under repair. The standby machine has three states: $\xi_s(t) = 1$ if the standby machine $M_s$ is operational, $\xi_s(t) = 2$ if the standby machine $M_s$ is under repair and $\xi_s(t) = 3$ if the standby machine $M_s$ is at time-off. Hence, the dynamics for a manufacturing system consisting of three machines (two machines with passive redundancy and one machine in series with the previous ones) is in a hybrid state comprising a continuous state and a discrete state $\xi(t) = (\xi_1(t), \xi_2(t), \xi(t))$ as follows:

### 3.1 Continuous state

We denote the number of parts in the WIP as $x_1(t)$ and the difference between cumulative production and demand as $x_2(t)$. Note that the control problem is subject to non-negative constraints on WIP, meaning that $x_1(t) \geq 0$. The surplus $x_2(t)$ can be positive (i.e. inventory costs $c^+_2$ are thus charged) or negative (i.e. backlog costs $c^-_2$ are thus charged).

We use $u_i(t)$ with $(i = 1, 2, s)$ to denote the input rate to $M_i$ and $x_1(t)$ to denote the number of parts in the buffer between $M_i$ and $M_2$ ($i = 1, s$).

The dynamics of the system can be written as follows:

$$\dot{x}_1(t) = \tilde{u}_1(t) - \tilde{u}_2(t) = u_1(t)I_1^p + u_s(t)I_s^p - u_2(t), \quad x_1(0) = x_1$$  \hspace{1cm} (1)

with:

$$I_1^p = \begin{cases} 1 & \text{if } \alpha = 3, 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad I_s^p = \begin{cases} 1 & \text{if } \alpha = 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $I_1^p$ and $I_s^p$ present the state of main machines and standby machine in each mode.

$$\dot{x}_2(t) = \tilde{u}_2(t) - \tilde{u}_3(t) = u_2(t) - u_3(t), \quad x_2(0) = x_2$$  \hspace{1cm} (2)

with:

$$u_3(t) := d$$

In matrix notation, the system of Equations (1) and (2) becomes:

$$\dot{x}(t) = A\tilde{u}(t), \quad x(0) = x$$  \hspace{1cm} (3)

where

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad \tilde{u}(t) = (\tilde{u}_1(t), \tilde{u}_2(t), \tilde{u}_3(t)) = (u_1(t)I_1^p + u_s(t)I_s^p, u_2(t), u_3(t))$$

Figure 1. Transfer line with passive redundancy producing one part type.
and

\[ x(t) = (x_1(t), x_2(t)) \]

with

\[ I_1^\alpha = \begin{cases} 1 & \text{if } \alpha = 3, 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases}, \quad I_s^\alpha = \begin{cases} 1 & \text{if } \alpha = 1, 2 \text{ and } u_s(t) := d. \end{cases} \]

### 3.2 Discrete state

Consistent with assumptions set out in Section 2.1, the operational mode of the whole system can be described by a random vector \( \xi(t) = (\xi_1(t), \xi_2(t), \xi_3(t)) \) taking values in \( \mathbf{B} = \{1, 2, 3, 4, 5, 6, 7, 8\} \). Without loss of generality, for the three-machine flow-shop case with passive redundancy, \( \xi(t) \) can be expressed as follows:

\[
\xi(t) = \begin{cases} 1 & \text{if } M_1 \text{ is under repair, } M_2 \text{ is operational and } M_s \text{ is operational}; \\ 2 & \text{if } M_1 \text{ is under repair, } M_2 \text{ is under repair and } M_s \text{ is operational}; \\ 3 & \text{if } M_1 \text{ is operational, } M_2 \text{ is operational and } M_s \text{ is under repair}; \\ 4 & \text{if } M_1 \text{ is operational, } M_2 \text{ is under repair and } M_s \text{ is under repair}; \\ 5 & \text{if } M_1 \text{ is under repair, } M_2 \text{ is operational and } M_s \text{ is under repair}; \\ 6 & \text{if } M_1 \text{ is under repair, } M_2 \text{ is under repair and } M_s \text{ is under repair}; \\ 7 & \text{if } M_1 \text{ is operational, } M_2 \text{ is operational and } M_s \text{ is at time-off}; \\ 8 & \text{if } M_1 \text{ is operational, } M_2 \text{ is under repair and } M_s \text{ is at time-off}. 
\]

Figure 2 displays the modes of the system associated with the process \( \xi(t) \).

The transition rate matrix of the stochastic processes \( \xi(t) \) is denoted by \( Q \) such that \( Q = \{q_{\alpha\beta}\} \), with \( q_{\alpha\beta} > 0 \) if \( \alpha \neq \beta \) and \( q_{\alpha\alpha} = \sum_{\beta \neq \alpha} q_{\alpha\beta} \), where \( \alpha, \beta \in \mathbf{B} \).

The transition probabilities associated to \( q_{\alpha\beta} \) are expressed as:

\[
p(\xi(t + \delta t) = \beta | \xi(t) = \alpha) = \begin{cases} q_{\alpha\beta}(\delta t + 0(\delta t)) & \text{if } \alpha \neq \beta, \\ 1 + q_{\alpha\beta}(\delta t + 0(\delta t)) & \text{if } \alpha = \beta. \end{cases}
\]

The transition rate matrix \( Q \) is expressed as follows:

\[
Q = \begin{bmatrix}
q_{11} & q_{12} & 0 & 0 & q_{15} & 0 & q_{17} & 0 \\
q_{21} & q_{22} & 0 & 0 & 0 & q_{26} & 0 & q_{28} \\
0 & 0 & q_{33} & q_{34} & q_{35} & 0 & q_{37} & 0 \\
0 & 0 & q_{43} & q_{44} & 0 & q_{46} & 0 & q_{48} \\
q_{51} & 0 & q_{53} & 0 & q_{55} & q_{56} & 0 & 0 \\
0 & q_{62} & 0 & q_{64} & q_{65} & q_{66} & 0 & 0 \\
q_{71} & 0 & q_{73} & 0 & 0 & 0 & q_{77} & q_{78} \\
0 & q_{82} & 0 & q_{84} & 0 & 0 & q_{87} & q_{88}
\end{bmatrix}.
\]

The set of admissible decisions at mode \( \alpha(t) \) and control policies (control variables) at mode \( \alpha(t) \):

\[
\Gamma(x, \alpha) = \left\{(u_1(\cdot), u_2(\cdot), u_s(\cdot)) \in \mathbb{R}^3, 0 \leq u_1(\cdot) \leq u_{1\text{max}}, 0 \leq u_2(\cdot) \leq u_{2\text{max}}, 0 \leq u_s(\cdot) \leq u_{s\text{max}}\right\}
\]

In Equation (6), \( u_{1\text{max}} \) is the main machine’s \( M_1 \) maximal production rate, \( u_{2\text{max}} \) is the main machine’s \( M_2 \) maximal production rate, and \( u_{s\text{max}} \) is the standby machine’s \( M_s \) maximal production rate. \( \Gamma(x, \alpha) \) denotes the set of all
admissible controls with respect to \( x \in \Lambda \) and \( \alpha(0) = \alpha \). Let \( \Lambda = [0, \infty) \times R \subset R^n \) denote the state constraint domain.

The control problem involves finding an admissible control law \( u(\cdot) = (u_1, u_2, u_r) \) that minimises the cost function \( J(\cdot) \) given by:

\[
J(\alpha, x, u) = E\left\{ \int_0^\infty e^{-\rho t} g(\alpha, x, \cdot)dt \mid x(0) = x, \xi(0) = \alpha \right\},
\]

where \( \rho \) is the discount rate and \( g(x, \alpha, \cdot) = c_1^+ x_1^+ + c_2^+ x_2^+ + c_2^- x_2^- + c^\alpha \) is the instantaneous cost, \( c_1^+, c_2^- \), and \( c^\alpha \), being the cost per unit to produce parts for inventory, backlog as well as intervention cost on the machine, respectively.

\[
x^+ = \max\{0, x\}, \quad x^- = \max\{-x, 0\}
\]

and

\[
c^\alpha = ((c_r + c_{\text{lagout}})u_r)\text{Ind}\{\alpha = 1\} + ((c_r + c_{\text{lagout}})u_r + (c_r + c_{\text{lagout}})u_r)\text{Ind}\{\alpha = 2\}
+ ((c_r + c_{\text{lagout}})u_r)\text{Ind}\{\alpha = 3\} + ((c_r + c_{\text{lagout}})u_r + (c_r + c_{\text{lagout}})u_r)\text{Ind}\{\alpha = 4\}
+ ((c_r + c_{\text{lagout}})u_r + (c_r + c_{\text{lagout}})u_r)\text{Ind}\{\alpha = 5\} + ((c_r + c_{\text{lagout}})u_r + (c_r + c_{\text{lagout}})u_r)
+ (c_r + c_{\text{lagout}})u_r)\text{Ind}\{\alpha = 6\} + ((c_r + c_{\text{lagout}})u_r + (c_r + c_{\text{lagout}})u_r)\text{Ind}\{\alpha = 8\}
\]
with

$$\text{Ind}\{\Theta(\cdot)\} = \begin{cases} 1 & \text{if } \Theta(\cdot) \text{ is true} \\ 0 & \text{otherwise} \end{cases}$$

for a given proposition \(\Theta(\cdot)\). The cost of corrective maintenance with logout/tagout depends on the duration of lockout/tagout and repair activities. This cost is described for main machine \(M_1\) by \((c_r + c_{\text{tagout}})u_1\), for the main machine \(M_2\) by \((c_r + c_{\text{tagout}})u_2\) and for the standby machine \(M_3\) by \((c_r + c_{\text{tagout}})u_3\).

Let \(v(x, \alpha)\) denote the value function or minimum discounted cost for Equations (7) as expressed in the following equation:

$$v(x, \alpha) = \min_{u \in \Gamma(x, \alpha)} J(\alpha, x, u), \quad \forall \alpha \in B. \quad (8)$$

In Appendix A, we present the properties of the value function \(v(\cdot)\) given by Equation (8). It is shown that the value function \(v(\cdot)\) given by Equation (8) should satisfy a set of partial differential equations known as the HJB equations in terms of DD.

### 4. Numerical example and sensitivity analysis

Let us consider a transfer line with three non-identical machines (two machines with passive redundancy and one machine in series with the previous ones). The system capacity is described by an eight-Markov process with states \(\xi(t) \in B = \{1, 2, 3, 4, 5, 6, 7, 8\}\).

The discrete dynamic programming Equation (A6) in Appendix A gives the eight equations that are presented in Appendix B.

We use the following computational domain:

$$C_d^h = \{(x_1, x_2) : 0 \leq x_1 \leq 5; -5 \leq x_2 \leq 5\}.$$ 

The parameters for our case study appear in Table 1.

The results obtained for the control variables \(u_1, u_2,\) and \(u_3\) of a production line with a passive redundancy system are given in Figures 3–9 for illustration purposes.

Figure 3 shows that there is no need to produce the part with sufficient stock levels both in the WIP, described by \(x_1\), and in the stock of finished products, described by \(x_2\). In mode 1, the production rate of the main machine \((M_1)\) is described by \(u_1 = 0\). For small stock levels, the policy obtained properly defines the region in the domain \((x_1, x_2)\) where a maximal production rate is optimal. In Figure 4, we have a similar trend in the optimal production rate as in mode 1. In mode 3, the production rate of the standby machine \((M_3)\) is described by \(u_3 = 0\). Figure 5 illustrates the same policy as in modes 1 and 3, except that at mode 7, the standby machine is at time-off, meaning that the production rate of standby machine \(u_3 = 0\).

Figure 6 illustrates that thanks to a standby machine, we can produce the part for the WIP, as described by \(x_1\). In mode 2, the production rates of the main machines \(M_1\) and \(M_2\) are described by \(u_1 = u_2 = 0\) respectively. Figure 7 shows that the main machine \(M_1\) produces parts for the WIP described by \(x_1\). In mode 4, the production rates of main machine \(M_2\) and of standby machine \(M_3\) are described by \(u_2 = u_3 = 0\) respectively. We have the same policy in Figure 8 as in modes 2 and 4, except that at mode 8, the standby machine is at time-off, meaning that the standby machine’s production rate is \(u_3 = 0\). The production rate of the main machine \(M_2\) at mode 8 is described by \(u_2 = 0\).

Figure 9 shows that the main machine \(M_2\) produces the finished products described by \(x_2\). In this mode, the production rates of main machine \(M_1\) and of standby machine \(M_3\) are described by \(u_1 = u_3 = 0\).

### Table 1. Parameters for the numerical example.

<table>
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<th>Parameter</th>
<th>(c_1^h)</th>
<th>(c_2^h)</th>
<th>(c_2)</th>
<th>(q_{12}^h)</th>
<th>(q_{12}^l)</th>
<th>(u_1)</th>
<th>(u_2)</th>
<th>(u_3)</th>
<th>(\mu_{10}^\text{max})</th>
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<td>0.08</td>
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<tr>
<td>Parameter</td>
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<td>(\mu_{20}^\text{max})</td>
<td>(c_{\text{tagout}})</td>
<td>(u_1)</td>
<td>(u_2)</td>
<td>(u_3)</td>
<td>(\rho)</td>
<td>(d)</td>
<td></td>
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<tr>
<td>Value</td>
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<td>150</td>
<td>200</td>
<td>250</td>
<td>0.2</td>
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</table>
Figure 3. Production rate of (a) $M_1$ and (b) $M_2$ at mode 1.

Figure 4. Production rate of (a) $M_1$ and (b) $M_2$ at mode 3.

Figure 5. Production rate of (a) $M_1$ and (b) $M_2$ at mode 7.
These results also illustrate that the main machines and standby machine are unavailable at mode 6, which means that \( u_1 = u_2 = u_s = 0 \).

The results illustrated in Figures 3–9 allowed us to determine our policy as follows:

\[
\begin{align*}
  u_i(x_1, x_2, 3) &= \begin{cases} 
  u_i^{\text{max}} & \text{if } x_1 < Z_1 \land x_2 < Z_2 \\
  d & \text{if } x_1 = Z_1 \land x_2 = Z_2 \\
  0 & \text{otherwise}
  \end{cases} \\
  u_2(x_1, x_2, 3) &= \begin{cases} 
  u_2^{\text{max}} & \text{if } x_2 < Z_3 \land x_1 > 0 \\
  d & \text{if } x_2 = Z_3 \land x_1 > 0 \\
  0 & \text{otherwise}
  \end{cases}
\end{align*}
\]  

With \( i = 1, S \)

Note that the aim of our policy is to find the optimal value of production rates \( u_i \) with \( i = 1, S \), which are dependent on two factors, \( Z_1 \) and \( Z_2 \), and the optimal value of production rate \( u_2 \), which is dependent on one factor such as \( Z_3 \), as illustrated in Figure 4.

The next sections are aimed at developing a systematic approach for determining the optimal values of \( Z_1, Z_2, Z_3 \).
5. Control approach, experimental design, and response surface methodology

In order to propose an approach that could be easily applied to control manufacturing systems at the operational level, the descriptive capacities of discrete event simulation models are combined with analytical models, experimental design, and response surface methodology. Many studies have been covered in the research literature on these subjects. For more details, we refer the reader to Gharbi and Kenne (2000). The structure of the proposed control approach is presented in Figure 10.

(1) The manufacturing system’s control problem statement, as shown in Section 3, consists of a production problem presentation for a transfer line with passive redundancy. This problem is presented through a stochastic optimal control model based on control theory. The aim of this step is to find the control variables \( (u_1(\cdot), u_2(\cdot), u_3(\cdot)) \), called the production rates. The control variables allow an improvement in the incurred cost.

(2) The optimality conditions, described by the HJBDD equations, are obtained from the problem statement of the first step. This step shows that the value function, representing the incurred cost, is the solution of the HJBDD equations, and our control policy (production rates) is near-optimal.

(3) In this step, we use numerical methods to solve the optimality equations of the problem because it is not possible to solve them analytically.

(4) The control factors \( Z_1, Z_2, Z_3 \) for the control production rates describe the numerical control policy obtained.

(5) The simulation model uses the near-optimal control policy defined in the previous step as the input factor for conducting experiments in order to evaluate the transfer line’s with passive redundancy performances. Therefore, the cost incurred is obtained for the given values of the control factors thanks to the simulation model, which will be presented in the next section.

(6) The experimental design approach defines how control factors can be varied in order to identify the effects of the main factors and their interactions on the cost. These variations must be evaluated through a minimal set of simulation experiments.

(7) In this step, we use a response surface methodology to obtain the relationship between the significant main factors and the incurred cost as well as the relationship between the main factors’ interactions. Thereafter, the optimised model obtained in order to determine the main factors’ best values are called \( Z_1^*, Z_2^*, Z_3^* \) for the production.

(8) The proposed control approach gives the production rates described by Equation (9), for the best values of factors \( Z_1, Z_2, Z_3 \) meaning \( Z_1^*, Z_2^*, Z_3^* \).

The performance-estimation tool chosen for this study is a discrete simulation model, and so we used the Arena software, which uses the SIMAN language; we refer the reader to Rossetti (2010).

The SIMAN portion of the software is composed of various networks describing specific tasks (failure and repair events, threshold production crossing of inventory variables, etc.). The simulation model is presented in Figure 11.

(1) The INITIALISATION block initialises the problem variables (current surplus, production rates, incurred cost, etc.).

(2) The DEMAND ARRIVAL block performs the arrival of a demand for each \( d \) unit of time. A verification test is then performed on the product’s inventory level, and the inventory or the backorder is updated.

(3) The CONTROL POLICY block is defined in Equation (9). The feedback control policy is defined by the output of the SIGNAL block, which is used to permanently verify the variation in the stock level \( x(t) \) in order to specify the best action to carry out.

(4) The STARVATION of the machines is implemented with the use of observation networks. Whenever the in-process buffer becomes empty, a signal is sent. Another signal is sent when the material becomes available for operation.

(5) The FAILURES AND REPAIRS block performs two functions: it defines the time-to-failure of the machine as well as the time-to-repair of the machine.

(6) The PARTS PRODUCTION block performs the production of finished products according to the policy defined by the CONTROL POLICY.

(7) The UPDATE INVENTORY block is used once the time step is chosen. For more details we refer the reader to Pritsker and O’Reilly (1999).
The UPDATE INCURRED COST block calculates the incurred cost according to the different variable levels and unit costs $c^+$ and $c^-$. The simulation ends when the current simulation time $t$ reaches the defined simulation period $T$. We thus ran offline simulations to determine the time necessary for the manufacturing system to reach its steady state. We found the simulation time for our manufacturing system at nearly 20,000 units of time, and this duration is used for all the simulations.

The simulation parameters used in this paper are the same as in Table 1. The input–output data are generated by the simulation model from the variation of independent factors $Z_1$, $Z_2$, and $Z_3$. Now, we present a procedure to vary these factors simultaneously. Such a procedure uses the experimental design approach. Hence, for three factor problems, as illustrated in the previous sections, we selected a $3^3$ response surface design, since we have three independent variables at three levels. This design leads to the completion of 27 experimental trials. The levels of the independent variables $Z_1$, $Z_2$, $Z_3$ range from a low of 5 to a high of 25. In this paper, we chose three replications, and thus have 108 $(27 \times 4)$ simulation runs. We refer the reader to Montgomery (2005) for more details. We considered all possible combinations of different levels of independent variables by response surface design. The objective of this design is to understand the effects of independent variables on performance measures, in our case, the production average cost. From the ANOVA table, the independent variables $Z_1$, $Z_2$, $Z_3$ and interaction effect as well as the quadratic effect are significant for the dependent variable at 0.05 level of significance. The $R^2$ value of 0.88, meaning 88% of the total variability, is explained by the model.

The average cost function is given by:

$$\text{Average cost} = \beta_0 + \sum_{i=1}^{n} \beta_i Z_i + \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} Z_i Z_j.$$  \hspace{1cm} (10)

The estimation of the regression coefficients is performed and 10 coefficients obtained in Table 2.

Table 2. Polynomial coefficients.

<table>
<thead>
<tr>
<th>$\beta_0$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_{11}$</th>
<th>$\beta_{12}$</th>
<th>$\beta_{13}$</th>
<th>$\beta_{22}$</th>
<th>$\beta_{23}$</th>
<th>$\beta_{33}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>860</td>
<td>1.899</td>
<td>–18.0</td>
<td>4.080</td>
<td>–0.086</td>
<td>–0.351</td>
<td>0.530</td>
<td>0.272</td>
<td>–0.153</td>
<td></td>
</tr>
</tbody>
</table>

$Z_1$, $Z_2$, $Z_3$ are the optimal values of independent variables $Z_1$, $Z_2$ and $Z_3$. Figures 12–14 illustrate the contour plots of the average cost function or response surface. These values determine the extension of the hedging point policy for the manufacturing system considered, where the average cost is minimised and this control policy is the best approximation of the optimal control. We observe that the cost function is not very sensitive to small variations of finished goods stock levels. The WIP stock
levels appear to be less sensitive to increases or decreases in WIP stock levels’ small variations. Because we can produce parts for WIP at all times thanks to passive redundancy, we can therefore respond to demand permanently.

To illustrate the effect of the variation inventory and backlog costs on the design parameters, a sensitivity analysis was conducted in Table 3, with $c_1 = 1$, $c_{r_1} = 150$, $c_{r_2} = 200$, $c_{r_3} = 250$ and $c_{\text{lagout}} = 50$.

The first section of Table 3 presents the variations in stock levels based on inventory costs, while the second section shows the variations in stock levels based on backlog costs.

Figures 15 and 16 plot variations in stock levels based on inventory costs and backlog costs, which are presented in Table 3. In these figures, homogenous variation occurs with passive redundancy, because it produces parts for WIP at all times and makes possible to respond to demand permanently. Therefore, we observe that variations in inventory and backlog cost do not influence the production thresholds of WIP and finished goods.

Production of a transfer line’s optimal costs was determined using the analytical model presented in this paper. The numerical approach, the experimental design, and the response surface methodology consecutively show that the resulting policy is optimal and enhances machine availability. Without loss of generality of this proposal, this
model is based on certain assumptions relating to a transfer line consisting of three machines that are not identical
and operate with passive redundancy. We observed that the passive redundancy case allows us to better optimise the
production cost of transfer lines while guaranteeing occupational safety. The integration of the second machine as
the passive redundancy allows demand to be met permanently. Furthermore, this integration will release the
intervention time needed for the machine which is down. It also minimises the possibility of circumvention of device
protection or retraction of lockout-tagout procedures.

6. Conclusion

This paper confirms that it is possible to integrate a passive redundancy system in a production line in order to:
(1) increase the productivity of workers and material resources and (2) better optimise production costs while
guaranteeing the safety of workers. In this paper, the control policy has an extension of hedging point structure.

![Contour plot of the response surface.](image)

Table 3. Variations in optimal design factors based on inventory and backlog costs.

<table>
<thead>
<tr>
<th>$c_1^*$</th>
<th>$c_2^*$</th>
<th>$Z_1^*$</th>
<th>$Z_2^*$</th>
<th>$Z_3^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>60</td>
<td>11.3</td>
<td>14.5</td>
<td>13.1</td>
</tr>
<tr>
<td>15</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>20</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>25</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>30</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>35</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>40</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>45</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>50</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>55</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>10</td>
<td>20</td>
<td>11.3</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>10</td>
<td>30</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>10</td>
<td>60</td>
<td>11.4</td>
<td>14.5</td>
<td>13.0</td>
</tr>
<tr>
<td>10</td>
<td>70</td>
<td>11.4</td>
<td>14.5</td>
<td>13.1</td>
</tr>
<tr>
<td>10</td>
<td>80</td>
<td>11.4</td>
<td>14.5</td>
<td>13.1</td>
</tr>
<tr>
<td>10</td>
<td>90</td>
<td>11.4</td>
<td>14.5</td>
<td>13.1</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>11.4</td>
<td>14.5</td>
<td>13.1</td>
</tr>
<tr>
<td>10</td>
<td>110</td>
<td>11.4</td>
<td>14.5</td>
<td>13.1</td>
</tr>
</tbody>
</table>
Based on the numerical solution obtained, a parameterised near-optimal control policy was derived. Such a policy depends on stock threshold levels. An experimental design was used to determine the effects of the independent variables on the average cost over the production horizon. We combined an analytical, simulation, and statistical method to provide the average cost estimation related to the control problem. The average cost estimation allows the best values for the control parameters to be determined. Finally, passive redundancy systems improve
production costs and worker safety. Finally, passive redundancy system improves production costs and worker safety. The former is obtained by meeting the demand whereas the latter is fulfilled by releasing essential space-time for the machines that are under repair.

Acknowledgements

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References


Appendix A: Optimal conditions and numerical approach

The properties of the value functions and the HJB in terms of DD for inner and boundary points are presented in this section. These equations describe the optimality conditions for a transfer line with passive redundancy producing one part type. Hence, we first present the notion of these derivatives and some related properties of convex functions.
A function \( f(x), x \in \mathbb{R}^n \), is said to have a directional derivative \( f'_p(x) \) along direction \( p \in \mathbb{R}^n \), if there exists
\[
\lim_{\delta \to 0} \frac{f(x + \delta p) - f(x)}{\delta} = f'_p(x).
\]
If a function \( f(x) \) is differentiable at \( x \), then \( f'_p(x) \) exists for every \( p \) and
\[
f'_p(x) = (\nabla f(x), p)
\]
where \( \nabla f(x) \) is the gradient of \( f(x) \), and \((.,.)\) is the scalar product. Furthermore, a continuous convex function defined on a convex domain \( \Sigma \) is differentiable almost everywhere, and has a directional derivative along any direction at any inner point of \( \Sigma \) and along any admissible direction (i.e. a direction \( p \) such that \( x + \delta p \in \Sigma \) for some \( \delta > 0 \)) at any boundary point of \( \Sigma \) (for more details, see Sethi and Zhang 1994).

Note that \( \{ \tilde{u}i : (u_1, u_2, u_3) \in \Gamma(x, \alpha) \} \) is the set of admissible directions at \( x \).

Regarding the optimality principle, let us write the set of partial differential equations known as the HJB equations in terms of DD as follows:
\[
\rho v(x, \alpha) = \min_{u_1, u_2, u_3} \left\{ v'_\alpha(x, \alpha) + g(\alpha, x, \cdot) + \sum_{\alpha \neq \beta} q_{\alpha \beta} [v(\alpha, \beta) - v(x, \alpha)] \right\}, \quad \forall \alpha, \beta \in B
\]
where:
\[
v'_\alpha(x, \alpha) = (\tilde{u}_1 - \tilde{u}_2)r_\alpha(x, \alpha) + (\tilde{u}_2 - \tilde{u}_3) \times r_\alpha(x, \alpha).
\]
With:
\[
\tilde{u}_1 = u_1I_1^\alpha + u_2I_2^\alpha, \quad I_1^\alpha = \begin{cases} 1 & \text{if } \alpha = 3, 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases}, \quad I_2^\alpha = \begin{cases} 1 & \text{if } \alpha = 1, 2 \\ 0 & \text{otherwise} \end{cases}, \quad \tilde{u}_2 = u_2 \quad \text{and} \quad \tilde{u}_3 = u_3 := d.
\]
The choice of standby machine and the main machine characteristics must be in such a way that respects the feasibility of system.

The system is considered feasible if:
\[
\sum \pi_i u_i^{\text{max}} \geq d, \quad (A2)
\]
where the limitation probabilities can be ascertained from the following equation for a system conforming to a Markov process:
\[
\pi(\cdot)Q(\cdot) = 0, \quad \sum_{i=1}^{n} \pi_i = 1
\]
with:

- \( \pi(\cdot) \) Limiting probabilities
- \( Q(\cdot) \) Transition matrix rates.

Hence, we have \( \pi(\cdot) = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6, \pi_7, \pi_8) \) representing the vector of limiting probabilities from modes 1 to 8.

Let \( \delta \Lambda \) denote the boundary of \( \Lambda \). If there exists \( x_i = 0 \) (i = 1, 2), then \( x \in \delta \Lambda \). Let the restriction of \( v(x, \alpha) \) on some \( j \)-dimensional face, \( 0 < j < m \), of \( \delta \Lambda \) be differentiable at an inner point \( x_0 \) of this face. Hence, there is a vector \( \nabla v(x_0, \alpha) \) such that \( v'_\alpha(x_0, \alpha) = (\nabla v(x_0, \alpha), p) \) for any admissible direction at \( x_0 \).

Now, we can write the boundary condition on \( v(., .) \) from the continuity of the value function by:
\[
\min_{u \in \Gamma(x_0, \alpha)} \left\{ (\nabla v(x_0, \alpha), \tilde{u}i) + g(\alpha, x_0, \cdot) + \sum_{\alpha \neq \beta} q_{\alpha \beta} [v(\alpha, \beta) - v(x_0, \alpha)] \right\} = \min_{u \in \Gamma(x_0)} \left\{ (\nabla v(x_0, \alpha), \tilde{u}i) + g(\alpha, x_0, \cdot) + \sum_{\alpha \neq \beta} q_{\alpha \beta} [v(\alpha, \beta) - v(x_0, \alpha)] \right\}, \quad \forall \alpha, \beta \in B
\]

We refer the reader to Lou and Zhang (1994) for the interpretation of the condition (A3).

The optimal control policy \( (u_1^*, u_2^*, u_3^*) \) denotes a minimiser over \( \Gamma(\alpha) \) of the right-hand side of Equation (A1). This policy is consistent with the value function obtained in Equation (8). The optimal control policy therefore rests in solving Equation (A1). Obtaining an analytical solution of equation (A1) is roughly impossible. The numerical solution of the HJB Equation (A1) in terms of DD is a challenge considered insurmountable in the scientific literature.

Now, we use numerical methods to solve the optimality conditions presented in this section. This method is based on Kushner’s approach (Kushner and Dupuis 1992). The basic idea behind this approach consists of using an approximation
scheme for the directional derivative of the value function $v(x, \alpha)$. Let $h_1$ and $h_2$ denote the length of the finite difference interval of the variables $x_1$ and $x_2$. Hence, using $h_1$ and $h_2$, $v(x, \alpha)$ is approximated by $\hat{v}(x, \alpha)$, and $v_{x_1}$ and $v_{x_2}$ are approximated by:

$$v_{x_1}(\cdot) = v'_{\tilde{u}_1-\tilde{u}_2}(\cdot) = \begin{cases} \frac{1}{h_1}(v^h(x_1, x_1 + h_1, \alpha) - v^h(x_1, \alpha)) \ast (\tilde{u}_1 - \tilde{u}_2) & \text{if } (\tilde{u}_1 - \tilde{u}_2) \geq 0 \\ \frac{1}{h_1}(v^h(x_1, \alpha) - v^h(x_1, x_1-h_1, \alpha)) \ast (\tilde{u}_1 - \tilde{u}_2) & \text{otherwise} \end{cases}$$

with:

$$\tilde{u}_1 = u_1 l^u_i + u_i l^v_i, \quad l^u_i = \begin{cases} 1 & \text{if } \alpha = 3, 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases}, \quad \Gamma_i = \begin{cases} 1 & \text{if } \alpha = 1, 2 \\ 0 & \text{otherwise} \end{cases} \text{ and } \tilde{u}_2 = u_2$$

(A4)

$$v_{x_2}(\cdot) = v'_{\tilde{u}_1-\tilde{u}_2}(\cdot) = \begin{cases} \frac{1}{h_2}(v^h(x_1, x_2 + h_2, \alpha) - v^h(x_1, \alpha)) \ast (\tilde{u}_2 - \tilde{u}_3) & \text{if } (\tilde{u}_2 - \tilde{u}_3) \geq 0 \\ \frac{1}{h_2}(v^h(x_1, \alpha) - v^h(x_1, x_2-h_2, \alpha)) \ast (\tilde{u}_2 - \tilde{u}_3) & \text{otherwise} \end{cases}$$

with:

$$\tilde{u}_2 = u_2 \quad \text{and} \quad \tilde{u}_3 = u_3 := d$$

(A5)

We manipulated the approximation arrived at in Equations (A4) and (A5) to rewrite the HJBDD Equations (A1) as follows:

$$\hat{v}(x, \alpha) = \min_{u \in \Gamma^h(x, \alpha)} \left\{ \left( \rho + |q_{\alpha\omega}| \right) + \left( \frac{(\tilde{u}_1 - \tilde{u}_2)}{h_1} + \frac{(\tilde{u}_2 - \tilde{u}_3)}{h_2} \right) \right\} \left\{ \sum_{p \neq \alpha} q_{\alpha \beta}(v^h(x, \beta)) + g(x, \cdot) \right\}$$

with:

$$\tilde{u}_1 = u_1 l^u_i + u_i l^v_i, \quad l^u_i = \begin{cases} 1 & \text{if } \alpha = 3, 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases}, \quad \Gamma_i = \begin{cases} 1 & \text{if } \alpha = 1, 2 \\ 0 & \text{otherwise} \end{cases} \text{ and } \tilde{u}_2 = u_2 \quad \text{and} \quad \tilde{u}_3 = u_3 := d$$

where $\Gamma^h(x, \alpha)$ is the discrete feasible control space, and the other terms used in Equation (A5) are defined as:

$$K^+_i = \begin{cases} 1 & \text{if } (\tilde{u}_1 - \tilde{u}_2) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad K^-_i = \begin{cases} 1 & \text{if } (\tilde{u}_1 - \tilde{u}_2) < 0 \\ 0 & \text{otherwise} \end{cases}$$

with:

$$\tilde{u}_1 = u_1 l^u_i + u_i l^v_i, \quad l^u_i = \begin{cases} 1 & \text{if } \alpha = 3, 4, 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases}, \quad \Gamma_i = \begin{cases} 1 & \text{if } \alpha = 1, 2 \\ 0 & \text{otherwise} \end{cases} \text{ and } \tilde{u}_2 = u_2$$

(A6)

$$K^+_i = \begin{cases} 1 & \text{if } (\tilde{u}_2 - \tilde{u}_3) \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad K^-_i = \begin{cases} 1 & \text{if } (\tilde{u}_2 - \tilde{u}_3) < 0 \\ 0 & \text{otherwise} \end{cases}$$

with:

$$\tilde{u}_2 = u_2 \quad \text{and} \quad \tilde{u}_3 = u_3 := d.$$
Appendix B

The discrete dynamic programming Equation (A6) in Appendix A gives the following eight equations:

\[
\begin{align*}
\nu^i(x,1) &= \min_{u \in \Gamma^i(x,1)} \left( \rho + \frac{|u_1 - u_2|}{h_1} + \frac{|u_2 - d|}{h_2} + q_{12} + q_{15} + q_{17} \right)^{-1} \begin{bmatrix}
|u_1 - u_2| & (\nu^i(x_1 + h_1, x_2, 1)k_1^+ + \nu^i(x_1 - h_1, x_2, 1)k_1^-) \\
1 & + \frac{u_2 - d}{h_2} (\nu^i(x_1 + x_2 + h_2, 1)k_2^+ + \nu^i(x_1 + x_2 - h_2, 1)k_2^-) \\
+ g(x, 1) + q_{12}^i \nu^i(x, 2) + q_{15} \nu^i(x, 5) + q_{17} \nu^i(x, 4)
\end{bmatrix}, \\
\nu^i(x, 2) &= \min_{u \in \Gamma^i(x,2)} \left( \rho + \frac{|u_1 - u_2|}{h_1} + \frac{d}{h_2} + q_{21} + q_{26} + q_{28} \right)^{-1} \begin{bmatrix}
|u_1 - u_2| & (\nu^i(x_1 + h_1, x_2, 2)k_1^+ + \nu^i(x_1 - h_1, x_2, 2)k_1^-) \\
1 & + \frac{u_2 - d}{h_2} (\nu^i(x_1 + x_2 + h_2, 2)k_2^+ + \nu^i(x_1 + x_2 - h_2, 2)k_2^-) \\
+ g(x, 2) + q_{21} \nu^i(x, 1) + q_{26} \nu^i(x, 6) + q_{28} \nu^i(x, 8)
\end{bmatrix}, \\
\nu^i(x, 3) &= \min_{u \in \Gamma^i(x,3)} \left( \rho + \frac{|u_1 - u_2|}{h_1} + \frac{|u_2 - d|}{h_2} + q_{34} + q_{35} + q_{37} \right)^{-1} \begin{bmatrix}
|u_1 - u_2| & (\nu^i(x_1 + h_1, x_2, 3)k_1^+ + \nu^i(x_1 - h_1, x_2, 3)k_1^-) \\
1 & + \frac{u_2 - d}{h_2} (\nu^i(x_1 + x_2 + h_2, 3)k_2^+ + \nu^i(x_1 + x_2 - h_2, 3)k_2^-) \\
+ g(x, 3) + q_{34} \nu^i(x, 4) + q_{35} \nu^i(x, 5) + q_{37} \nu^i(x, 7)
\end{bmatrix}, \\
\nu^i(x, 4) &= \min_{u \in \Gamma^i(x,4)} \left( \rho + \frac{|u_1 - u_2|}{h_1} + \frac{d}{h_2} + q_{43} + q_{46} + q_{48} \right)^{-1} \begin{bmatrix}
|u_1 - u_2| & (\nu^i(x_1 + h_1, x_2, 4)k_1^+ + \nu^i(x_1 - h_1, x_2, 4)k_1^-) \\
1 & \times \frac{d}{h_2} (\nu^i(x_1 + x_2 + h_2, 4)k_2^+ + \nu^i(x_1 + x_2 - h_2, 4)k_2^-) \\
+ g(x, 4) + q_{43} \nu^i(x, 3) + q_{46} \nu^i(x, 6) + q_{48} \nu^i(x, 8)
\end{bmatrix}, \\
\nu^i(x, 5) &= \min_{u \in \Gamma^i(x,5)} \left( \rho + \frac{u_2}{h_1} + \frac{|u_2 - d|}{h_2} + q_{51} + q_{53} + q_{56} \right)^{-1} \begin{bmatrix}
\frac{u_2}{h_1} (\nu^i(x_1 - h_1, x_2, 5)k_1^+ + \frac{u_2 - d}{h_2} (\nu^i(x_1 + h_1, x_2, 5)k_2^+) \\
\times g(x, 5) + q_{51} \nu^i(x, 1) + q_{53} \nu^i(x, 3) + q_{56} \nu^i(x, 6)
\end{bmatrix}, \\
\nu^i(x, 6) &= \min_{u \in \Gamma^i(x,6)} \left( \rho + \frac{d}{h_2} + q_{62} + q_{64} + q_{65} \right)^{-1} \begin{bmatrix}
\frac{d}{h_2} (\nu^i(x_1 + x_2 - h_2, 6)k_2^+) + g(x, 6) + q_{62} \nu^i(x, 2) \\
+ q_{64} \nu^i(x, 4) + q_{65} \nu^i(x, 5)
\end{bmatrix}, \\
\nu^i(x, 7) &= \min_{u \in \Gamma^i(x,7)} \left( \rho + \frac{|u_1 - u_2|}{h_1} + \frac{|u_2 - d|}{h_2} + q_{71} + q_{73} + q_{78} \right)^{-1} \begin{bmatrix}
|u_1 - u_2| & (\nu^i(x_1 + h_1, x_2, 7)k_1^+ + \nu^i(x_1 - h_1, x_2, 7)k_1^-) \\
1 & + \frac{u_2 - d}{h_2} (\nu^i(x_1 + x_2 + h_2, 7)k_2^+ + \nu^i(x_1 + x_2 - h_2, 7)k_2^-) \\
+ g(x, 7) + q_{71} \nu^i(x, 1) + q_{73} \nu^i(x, 3) + q_{78} \nu^i(x, 8)
\end{bmatrix}, \\
\nu^i(x, 8) &= \min_{u \in \Gamma^i(x,8)} \left( \rho + \frac{|u_1 - u_2|}{h_1} + \frac{d}{h_2} + q_{82} + q_{84} + q_{87} \right)^{-1} \begin{bmatrix}
|u_1 - u_2| & (\nu^i(x_1 + h_1, x_2, 8)k_1^+ + \nu^i(x_1 - h_1, x_2, 8)k_1^-) \\
1 & + \frac{d}{h_2} (\nu^i(x_1 + x_2 - h_2, 8)k_2^+) + g(x, 8) \\
+ q_{82} \nu^i(x, 2) + q_{84} \nu^i(x, 4) + q_{87} \nu^i(x, 7)
\end{bmatrix}.
\end{align*}
\]